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ON IMPLICATIONAL CLASSES OF STRUCTURES

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In the previous paper [1], we have studied that the least universal Horn class containing a given class K is constructed by taking all isomorphic copies of direct limits of substructures of direct products of structures in K . A universal Horn class may be also called a generalized implicational class. However, this generalized implicational class is restricted to being defined by a set of generalized implicational sentences of finite length.

In this paper, a (generalized) $L(\mathfrak{m}, \mathfrak{n})$ -implicational class will be defined by a set of (generalized) $L(\mathfrak{m}, \mathfrak{n})$ -implicational sentences each of which contains a conjunction of length $< \omega_{\mathfrak{n}}$ and a universal quantification over a string of variables of length $< \omega_{\mathfrak{m}}$ where $\mathfrak{m}, \mathfrak{n}$ are infinite cardinals, and $\omega_{\mathfrak{m}}, \omega_{\mathfrak{n}}$ are the initial ordinals of powers $\mathfrak{m}, \mathfrak{n}$ respectively. We shall make the similar investigation for a (generalized) $L(\mathfrak{m}, \mathfrak{n})$ -implicational class as in the above for a universal Horn class. The method of this study is analogous to that of the paper [1], but the results are not mere generalizations of the results in [1]. It can be seen from our results, especially from Theorem 1, that the lengths of conjunctions and quantifications are closely connected with direct limits and unions respectively. Theorem 2 is a direct generalization of the above result in [1]. The characterization (iii) of an $L(\mathfrak{m}, \mathfrak{n})$ -implicational class in the final

remark appears to make the substance of the first main theorem of the paper [3] clear, with the help of Theorem 2 in [2].

§ 1. Preliminaries.

Let L be a first order language with equality which has a set $\{v_\eta \mid \eta < \omega_{\mathfrak{m}}\}$ of individual variables, where $\omega_{\mathfrak{m}}$ is the initial ordinal of an infinite power \mathfrak{m} . All operation and relation symbols are assumed to be finitary. We use $x_0, x_1, \dots, x_\xi, \dots$ as syntactical variables which vary through the variables v_η , $\eta < \omega_{\mathfrak{m}}$, and denote by $(x_\xi \mid \xi < \rho)$ a subsequence of the sequence $(v_\eta \mid \eta < \omega_{\mathfrak{m}})$. A formula Φ of L which contains at most some of x_ξ , $\xi < \rho$, as free variables is denoted by $\Phi(x_\xi \mid \xi < \rho)$, if the variables x_ξ , $\xi < \rho$, need to be indicated. If ρ is finite, $\Phi(x_\xi \mid \xi < \rho)$ may be simply denoted by $\Phi(x_0, \dots, x_{\rho-1})$. An atomic formula of L means a formula of the form $t_1 = t_2$ or of the form $rt_1 \dots t_n$, where r is an n -ary relation symbol of L and t_1, \dots, t_n are terms of L . A structure \mathbb{A} of the similarity type corresponding to the language L is simply called a structure for L . The domain of \mathbb{A} is denoted by $D[\mathbb{A}]$. Let $\Phi(x_\xi \mid \xi < \rho)$ be a formula of L , and let $(a_\xi \mid \xi < \rho)$ be a ρ -sequence of elements in $D[\mathbb{A}]$. Then we write $(\mathbb{A}; (a_\xi \mid \xi < \rho)) \models \Phi(x_\xi \mid \xi < \rho)$, if $(a_\xi \mid \xi < \rho)$ satisfies $\Phi(x_\xi \mid \xi < \rho)$ in \mathbb{A} when the free variables x_ξ , $\xi < \rho$, are assigned the values a_ξ , $\xi < \rho$, respectively. If ρ is finite, $(\mathbb{A}; (a_\xi \mid \xi < \rho)) \models \Phi(x_\xi \mid \xi < \rho)$ may be denoted by $(\mathbb{A}; a_0, \dots, a_{\rho-1}) \models \Phi(x_0, \dots, x_{\rho-1})$.

Let \mathbb{A} and \mathbb{B} be structures for a language L . A mapping h of $D[\mathbb{A}]$ into (or onto) $D[\mathbb{B}]$ is called an L -homomorphism of \mathbb{A} into (or onto) \mathbb{B} , if for any atomic formula $\Theta(x_\xi \mid \xi < \rho)$ of

L and for any ρ -sequence $(a_\xi \mid \xi < \rho)$ of elements in $D[\mathbb{A}]$, $(\mathbb{A}; (a_\xi \mid \xi < \rho)) \models \theta(x_\xi \mid \xi < \rho)$ implies $(\mathbb{B}; (h(a_\xi) \mid \xi < \rho)) \models \theta(x_\xi \mid \xi < \rho)$. An L -homomorphism h of \mathbb{A} onto \mathbb{B} is called an L -isomorphism of \mathbb{A} onto \mathbb{B} , if the mapping h is one-to-one and the inverse mapping h^{-1} is also an L -homomorphism.

Let $(\mathbb{A}_i \mid i \in I)$ be a family of structures for L . A structure \mathbb{A} for L is called the direct product of the \mathbb{A}_i , $i \in I$, if the following two conditions hold:

- (1) $D[\mathbb{A}]$ is the Cartesian product $\prod(D[\mathbb{A}_i] \mid i \in I)$;
- (2) For any atomic formula $\theta(x_\xi \mid \xi < \rho)$ and any ρ -sequence $(a_\xi \mid \xi < \rho)$ of elements in $D[\mathbb{A}]$, $(\mathbb{A}; (a_\xi \mid \xi < \rho)) \models \theta(x_\xi \mid \xi < \rho)$ holds if and only if $(\mathbb{A}_i; (a_\xi(i) \mid \xi < \rho)) \models \theta(x_\xi \mid \xi < \rho)$ holds for all $i \in I$, where $a_\xi(i)$ denotes the i -th component of a_ξ .

The above definition of a direct product is equivalent to the usual definition of a direct product. Hence for any family of structures for L , the direct product of this family exists. From the above definition, the direct product of the empty family of structures for L is a one-element structure for L in which every atomic formula is valid. Such a structure is called an L -trivial structure.

X is called an operator if for every class K of structures for L , $X(K)$ is also a class of structures for L . If X and Y are operators, the operator XY is defined by $XY(K) = X(Y(K))$. The operators I , S , P , and P^* are defined as follows:

$I(K)$: all L -isomorphic copies of structures in K ;

$S(K)$: all substructures of structures in K ;

$P(K)$: all direct products of non-empty families of structures

in K ;

$P^*(K)$: all direct products of empty or non-empty families of structures in K .

Let E be a set of constant symbols (i.e. nullary operation symbols) not belonging to the language L . Then, a new first order language can be obtained from L by adjoining all the constant symbols e in E , which is denoted by $L(E)$. If $L(E)$ contains at least one constant symbol, then E is said to be L -generative. Now let \mathcal{A} be a structure for L , and ψ a mapping of E into $D[\mathcal{A}]$. Then \mathcal{A} can be expanded to a structure for $L(E)$ by considering $\psi(e)$ as realizations of e to \mathcal{A} . Such an expanded structure is denoted by $\mathcal{A}(\psi)$. An ordered pair (E, Ω) is called an L -defining pair, if E is an L -generative set of constant symbols not belonging to L and Ω is a set of atomic sentences of $L(E)$. For any infinite cardinals \mathfrak{p} and \mathfrak{q} , an L -defining pair (E, Ω) is called an $L(\mathfrak{p}, \mathfrak{q})$ -defining pair if $\overline{E} < \mathfrak{p}$ and $\overline{\Omega} < \mathfrak{q}$, where \overline{E} and $\overline{\Omega}$ denote the cardinals of E and Ω respectively.

Let K be a class of structures for a language L , and let (E, Ω) be an L -defining pair. Now let \mathcal{A} be a structure for L , and ψ a mapping of E into $D[\mathcal{A}]$. The pair (\mathcal{A}, ψ) is called a K -model of (E, Ω) , if \mathcal{A} is in K and every atomic sentence in Ω is valid in $\mathcal{A}(\psi)$. We denote by $(E, \Omega; K)$ the class of all K -models of (E, Ω) . A K -model of (E, Ω) , say (\mathcal{F}, ϕ) , is said to be free (in $(E, \Omega; K)$), if \mathcal{F} is generated by $\{\phi(e) \mid e \in E\}$ and for any $(\mathcal{A}, \psi) \in (E, \Omega; K)$, there exists an $L(E)$ -homomorphism of $\mathcal{F}(\phi)$ into $\mathcal{A}(\psi)$, i.e. there exists an L -homomorphism of \mathcal{F} into \mathcal{A} that maps $\phi(e)$ to $\psi(e)$ for

each $e \in E$. We denote by $F(E, \Omega; K)$ the class of all free K -models of (E, Ω) . Note that if (\mathbb{F}, ϕ) and (\mathbb{F}', ϕ') are in $F(E, \Omega; K)$, then $\mathbb{F}(\phi)$ and $\mathbb{F}'(\phi')$ are $L(E)$ -isomorphic.

The following criterion for a class K to possess free K -models can be immediately obtained from Theorem 2 in [1]:

CRITERION. Let K be a class of structures for a language L . Then, in order that for any L -defining pair $(E, \Omega), (E, \Omega; K) \neq \emptyset$ implies $F(E, \Omega; K) \neq \emptyset$, it is necessary and sufficient that $S(K) \subseteq I(K)$ and $P(K) \subseteq I(K)$.

§ 2. The definition of a (generalized) $L(\mathbb{m}, \mathbb{n})$ -implicational class and its simple properties.

Let \mathbb{m}, \mathbb{n} be any infinite cardinals, and let $\omega_{\mathbb{m}}, \omega_{\mathbb{n}}$ be the initial ordinals of powers \mathbb{m}, \mathbb{n} respectively. Let L be a first order language with equality which has a set $\{v_{\xi} \mid \xi < \omega_{\mathbb{m}}\}$ of variables. A new expression ----- which contains a conjunction of length $< \omega_{\mathbb{n}}$ and a quantification over a string of variables of length $< \omega_{\mathbb{m}}$ ----- of the form

$$(*) \quad \forall (x_{\xi} \mid \xi < \alpha) [\wedge (\theta_{\eta} \mid \eta < \beta) \rightarrow \theta]$$

is called a (generalized) $L(\mathbb{m}, \mathbb{n})$ -implicational sentence, if $\alpha < \omega_{\mathbb{m}}, \beta < \omega_{\mathbb{n}}$, and all θ_{η} and θ are (identically false or) atomic formulas of L which contain at most some of the variables $x_{\xi}, \xi < \alpha$. Note that every $L(\mathbb{m}, \mathbb{n})$ -implicational sentence is a generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentence.

Let \mathbb{A} be a structure for L . The sentence $(*)$ is said to be valid in \mathbb{A} , if for any α -sequence $(a_{\xi} \mid \xi < \alpha)$ of elements in $D[\mathbb{A}]$,

$$\begin{aligned} (\#) \quad & (\mathbb{A}; (a_{\xi} \mid \xi < \alpha)) \models \theta_{\eta}(x_{\xi} \mid \xi < \alpha) \text{ for all } \eta < \beta \text{ implies} \\ & (\mathbb{A}; (a_{\xi} \mid \xi < \alpha)) \models \theta(x_{\xi} \mid \xi < \alpha). \end{aligned}$$

Hence, if θ is an identically false formula, the condition (#) can be replaced by

(##) $(\mathbb{A}; (a_\xi \mid \xi < \alpha)) \models \neg \theta_\eta (x_\xi \mid \xi < \alpha)$ for some $\eta < \beta$.

Therefore the sentence (*) in this special case may be denoted by

$$\forall (x_\xi \mid \xi < \alpha) [\forall (\neg \theta_\eta \mid \eta < \beta)]$$

which contains a disjunction of length $< \omega_{(\mathfrak{n})}$. Let ϕ be a usual or generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational sentence of L . If ϕ is valid in a structure \mathbb{A} for L , then we write $\mathbb{A} \models \phi$.

Let Σ be a set of generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational sentences. A structure \mathbb{A} for L is called a model of Σ , if every sentence in Σ is valid in \mathbb{A} . The class of all models of Σ is denoted by Σ^* . A class K of structures for L is called a (generalized) $L(\mathfrak{m}, \mathfrak{n})$ -implicational class, if $K = \Sigma^*$ for some set Σ of (generalized) $L(\mathfrak{m}, \mathfrak{n})$ -implicational sentences. Note that every $L(\mathfrak{m}, \mathfrak{n})$ -implicational class is a generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational class.

The following lemmas can be easily obtained from the above definitions:

LEMMA 1. Let K be a generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational class. Then K is closed under the formation of substructures, i.e. $S(K) \subseteq K$.

LEMMA 2. Let K be a (generalized) $L(\mathfrak{m}, \mathfrak{n})$ -implicational class. Then K is closed under the formation of direct products of (non-empty) families of structures. That is, $P(K) \subseteq K$ for every generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational class K , especially $P^*(K) \subseteq K$ for every $L(\mathfrak{m}, \mathfrak{n})$ -implicational class K .

Let M be a partially ordered set, and let \mathfrak{p} be any infinite cardinal. M is said to be \mathfrak{p} -directed if for any subset N

of M which satisfies $\bar{N} < \mathbb{P}$, there exists a element $\mu \in M$ such that $v \leq \mu$ for all $v \in N$. A family $(\mathbb{A}_\mu \mid \mu \in M)$ of structures for L indexed by a set M is said to be \mathbb{P} -directed if M is an \mathbb{P} -directed partially ordered set and $\mathbb{A}_\mu \subseteq \mathbb{A}_\nu$ whenever $\mu \leq \nu$. Let $(\mathbb{A}_\mu \mid \mu \in M)$ be a \mathbb{P} -directed family of structures for L . A structure \mathbb{A} for L is called a union of $(\mathbb{A}_\mu \mid \mu \in M)$ and denoted by $\bigcup (\mathbb{A}_\mu \mid \mu \in M)$, if $D[\mathbb{A}] = \bigcup (D[\mathbb{A}_\mu] \mid \mu \in M)$ and each \mathbb{A}_μ is a substructure of \mathbb{A} . Let K be a class of structures for L . We denote by $U_{\mathbb{P}}(K)$ the class of all structures that are unions of \mathbb{P} -directed families of structures in K .

Now we shall prove the following:

LEMMA 3. Let K be a generalized $L(\mathbb{m}, \mathbb{n})$ -implicational class. Then K is closed under the formation of unions of \mathbb{m} -directed families of structures in K , i.e. $U_{\mathbb{m}}(K) \subseteq K$.

Proof. Let Σ be a set of generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentences such that $\Sigma^* = K$. Let $F = (\mathbb{A}_\mu \mid \mu \in M)$ be any \mathbb{m} -directed family of structures in Σ^* , and let \mathbb{A} be the union of F . Now let

$$\Phi = \forall (x_\xi \mid \xi < \alpha) [\wedge (\theta_n \mid n < \beta) \rightarrow \theta]$$

be any generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentence in Σ , and let $(a_\xi \mid \xi < \alpha)$ be any α -sequence of elements in $D[\mathbb{A}]$.

Now assume that

$$(\mathbb{A}; (a_\xi \mid \xi < \alpha)) \models \theta_n(x_\xi \mid \xi < \alpha) \text{ for all } n < \beta.$$

We shall prove that

$$(\mathbb{A}; (a_\xi \mid \xi < \alpha)) \models \theta(x_\xi \mid \xi < \alpha).$$

By the definition of a union, there exists a subfamily

$(\mathbb{A}_{\mu_\xi} \mid \xi < \alpha)$ of F such that $a_\xi \in D[\mathbb{A}_{\mu_\xi}]$ for each $\xi < \alpha$.

Hence there exists a structure $\mathbb{A}_\mu \in F$ such that $\mathbb{A}_{\mu_\xi} \subseteq \mathbb{A}_\mu$ for all $\xi < \alpha$, because $\alpha < \omega_{\mathbb{M}}$ and F is an \mathbb{M} -directed family.

Hence

$(\mathbb{A}_\mu; (a_\xi \mid \xi < \alpha)) \models \theta_\eta(x_\xi \mid \xi < \alpha)$ for all $\eta < \beta$,
because $\mathbb{A}_\mu \subseteq \mathbb{A}$ and $a_\xi \in D[\mathbb{A}_\mu]$ for all $\xi < \alpha$. Since $\mathbb{A}_\mu \in \Sigma^*$,
we have $\mathbb{A}_\mu \models \Phi$. Hence

$$(\mathbb{A}_\mu; (a_\xi \mid \xi < \alpha)) \models \theta(x_\xi \mid \xi < \alpha),$$

and hence

$$(\mathbb{A}; (a_\xi \mid \xi < \alpha)) \models \theta(x_\xi \mid \xi < \alpha).$$

Therefore every generalized $L(\mathbb{M}, \mathbb{N})$ -implicational sentence in Σ is valid in \mathbb{A} , i.e. $\mathbb{A} \in \Sigma^*$. This completes the proof.

Let $(\mathbb{A}_\mu \mid \mu \in M)$ be a family of structures for L indexed by a directed partially ordered set M , and let $(f_\mu^\nu \mid \mu, \nu \in M \text{ and } \mu \leq \nu)$ be a family of L -homomorphisms f_μ^ν of \mathbb{A}_μ into \mathbb{A}_ν such that f_μ^μ is the identity mapping for each $\mu \in M$ and $f_\mu^\nu f_\lambda^\mu = f_\mu^\nu$ whenever $\lambda \leq \mu \leq \nu$. Then the system $S = \langle (\mathbb{A}_\mu \mid \mu \in M), (f_\mu^\nu \mid \mu, \nu \in M \text{ and } \mu \leq \nu) \rangle$ is called a direct system. Let $A = \bigcup (D[\mathbb{A}_\mu] \times \{\mu\} \mid \mu \in M)$, and let \sim be the equivalence relation on A defined by

$$\langle a, \mu \rangle \sim \langle b, \nu \rangle \text{ if and only if for some } \lambda \in M, f_\mu^\lambda(a) = f_\nu^\lambda(b).$$

Now let \hat{A} be the set of all equivalence classes of A defined by the relation \sim . Then a structure $\hat{\mathbb{A}}$ for L is called a direct limit of the direct system S if the following two conditions hold:

- (1) $D[\hat{\mathbb{A}}] = \hat{A}$;
- (2) For any atomic formula $\theta(x_1, \dots, x_n)$ of L and for any elements $\hat{a}_1, \dots, \hat{a}_n$ in $D[\hat{\mathbb{A}}]$,
 $(\hat{\mathbb{A}}; \hat{a}_1, \dots, \hat{a}_n) \models \theta(x_1, \dots, x_n)$ if and only if there exist

some $\mu \in M$ and some elements a_1, \dots, a_n in $D[\hat{A}_\mu]$ such that $(\hat{A}_\mu; a_1, \dots, a_n) \models \theta(x_1, \dots, x_n)$ and $\langle a_i, \mu \rangle \in \hat{a}_i$ for each $i = 1, \dots, n$.

Note that the above definition of a direct limit is equivalent to the usual definition of a direct limit. Hence for any direct system S , there exists the direct limit of S .

Let \mathbb{P} be any infinite cardinal. A direct system $\langle (\hat{A}_\mu \mid \mu \in M), (f_\mu^\nu \mid \mu, \nu \in M \text{ and } \mu \leq \nu) \rangle$ is called a \mathbb{P} -direct system if the index set M is \mathbb{P} -directed. Let K be a class of structures for L . We denote by $L_{\mathbb{P}}(K)$ the class of all structures that are direct limits of \mathbb{P} -direct systems of structures in K .

LEMMA 4. Let K be a generalized $L(\mathbb{M}, \mathbb{N})$ -implicational class. Then K is closed under the formation of direct limits of \mathbb{N} -direct systems of structures in K , i.e. $L_{\mathbb{N}}(K) \subseteq K$.

Proof. Let Σ be a set of generalized $L(\mathbb{M}, \mathbb{N})$ -implicational sentences such that $\Sigma^* = K$. Let

$$S = \langle (\hat{A}_\mu \mid \mu \in M), (f_\mu^\nu \mid \mu, \nu \in M \text{ and } \mu \leq \nu) \rangle$$

be any \mathbb{N} -direct system of structures in Σ^* , and let \hat{A} be the direct limit of S . Now let

$$\Phi = \forall (x_\xi \mid \xi < \alpha) [\wedge (\theta_\eta \mid \eta < \beta) \rightarrow \theta_\beta]$$

be any generalized $L(\mathbb{M}, \mathbb{N})$ -implicational sentence in Σ , and let $(\hat{a}_\xi \mid \xi < \alpha)$ be any α -sequence of elements in $D[\hat{A}]$.

Now assume that

$$(\hat{A}; (\hat{a}_\xi \mid \xi < \alpha)) \models \theta_\eta(x_\xi \mid \xi < \alpha) \text{ for all } \eta < \beta.$$

We shall prove that

$$(\hat{A}; (\hat{a}_\xi \mid \xi < \alpha)) \models \theta_\beta(x_\xi \mid \xi < \alpha).$$

For each $\eta \leq \beta$, we define X_η as the sequence of ordinals such

that $\{x_\xi \mid \xi \in X_\eta\}$ is the set of all variables appearing in θ_η . Since $\hat{\mathbb{A}}$ is the direct limit of S , for each $\eta < \beta$, there exist an element $\mu_\eta \in M$ and a sequence $(a_\xi^{\mu_\eta} \mid \xi \in X_\eta)$ of elements in $D[\hat{\mathbb{A}}_{\mu_\eta}]$ such that

$$\begin{aligned} &(\hat{\mathbb{A}}_{\mu_\eta}; (a_\xi^{\mu_\eta} \mid \xi \in X_\eta)) \models \theta_\eta(x_\xi \mid \xi \in X_\eta), \text{ and} \\ &\langle a_\xi^{\mu_\eta}, \mu_\eta \rangle \in \hat{a}_\xi \text{ for each } \xi \in X_\eta. \end{aligned}$$

Moreover, there exist an element $\mu_\beta \in M$ and a sequence $(a_\xi^{\mu_\beta} \mid \xi \in X_\beta)$ of elements in $D[\hat{\mathbb{A}}_{\mu_\beta}]$ such that

$$\langle a_\xi^{\mu_\beta}, \mu_\beta \rangle \in \hat{a}_\xi \text{ for each } \xi \in X_\beta.$$

For each $\xi \in \bigcup(X_\eta \mid \eta \leq \beta)$, we now define Y_ξ as the set of all η such that $X_\eta \ni \xi$. Then for all $\eta \in Y_\xi$, $\langle a_\xi^{\mu_\eta}, \mu_\eta \rangle$ are in \hat{a}_ξ . Hence for any pair $\langle \eta, \eta' \rangle \in Y_\xi \times Y_\xi$, there exists an element $v_{\eta, \eta'} \in M$ such that

$$f_{\mu_\eta}^{v_{\eta, \eta'}}(a_\xi^{\mu_\eta}) = f_{\mu_{\eta'}}^{v_{\eta, \eta'}}(a_\xi^{\mu_{\eta'}}).$$

Since $\overline{Y_\xi \times Y_\xi} < \mathfrak{n}$ and M is \mathfrak{n} -directed, there exists an element $v_\xi \in M$ such that $v_{\eta, \eta'} \leq v_\xi$ for all $\langle \eta, \eta' \rangle \in Y_\xi \times Y_\xi$. Hence all $f_{\mu_\eta}^{v_\xi}(a_\xi^{\mu_\eta})$, $\eta \in Y_\xi$, are the same element in $D[\hat{\mathbb{A}}_{v_\xi}]$. Since each X_η is finite, $\overline{\bigcup(X_\eta \mid \eta \leq \beta)} < \mathfrak{n}$. Hence there exists an element $v \in M$ such that $v_\xi \leq v$ for all $\xi \in \bigcup(X_\eta \mid \eta \leq \beta)$. And hence for each element $\xi \in \bigcup(X_\eta \mid \eta \leq \beta)$,

$$\text{all } f_{\mu_\eta}^v(a_\xi^{\mu_\eta}), \eta \in Y_\xi, \text{ are the same element in } D[\hat{\mathbb{A}}_v].$$

Therefore for each $\xi \in \bigcup(X_\eta \mid \eta \leq \beta)$, we can define an element a_ξ in $D[\hat{\mathbb{A}}_v]$ by

$$a_\xi = f_{\mu_\eta}^v(a_\xi^{\mu_\eta}) \text{ for some } \eta \in Y_\xi.$$

Then we can immediately obtain the following:

$$(\hat{\mathbb{A}}_v; (a_\xi \mid \xi \in \bigcup(X_\eta \mid \eta \leq \beta))) \models \theta_\eta(x_\xi \mid \xi \in \bigcup(X_\eta \mid \eta \leq \beta))^{1)} \text{ for all } \eta < \beta, \text{ and } \langle a_\xi, v \rangle \in \hat{a}_\xi \text{ for each } \xi \in \bigcup(X_\eta \mid \eta \leq \beta).$$

Since $\hat{\mathbb{A}}_v \models \Phi$, we have

$$(\hat{A}_\beta; (a_\xi \mid \xi \in \bigcup (X_\eta \mid \eta \leq \beta))) \models \theta_\beta(x_\xi \mid \xi \in \bigcup (X_\eta \mid \eta \leq \beta))^2).$$

Hence by the definition of a direct limit, we have

$$(\hat{A}; (\hat{a}_\xi \mid \xi < \alpha)) \models \theta_\beta(x_\xi \mid \xi < \alpha),$$

as desired. Hence every generalized $L(\mathbb{M}, \mathbb{N})$ -implicational sentence in Σ is valid in \hat{A} , i.e. $\hat{A} \in \Sigma^*$. This completes the proof.

§ 3. Some lemmas concerning free structures and natural limit structures.

Let K be a class of structures for L , and let (E, Ω) be any L -defining pair. We denote by $M_{\mathbb{P}, \mathbb{Q}}^{(E, \Omega)}$ the set of all $L(\mathbb{P}, \mathbb{Q})$ -defining pairs (X, Γ) which satisfy $X \subseteq E$ and $\Gamma \subseteq \Omega$, where \mathbb{P} and \mathbb{Q} are infinite cardinals. For $(X, \Gamma), (Y, \Delta) \in M_{\mathbb{P}, \mathbb{Q}}^{(E, \Omega)}$, we define $(X, \Gamma) \leq (Y, \Delta)$ as both $X \subseteq Y$ and $\Gamma \subseteq \Delta$. Then $M_{\mathbb{P}, \mathbb{Q}}^{(E, \Omega)}$ forms a directed partially ordered set. Now assume that for each $(X, \Gamma) \in M_{\mathbb{P}, \mathbb{Q}}^{(E, \Omega)}$, $F(X, \Gamma; K) \neq \emptyset$, i.e. there exists $(\hat{A}_{(X, \Gamma)}, \phi_{(X, \Gamma)})$ in $F(X, \Gamma; K)$. Then, for all $(X, \Gamma), (Y, \Delta) \in M_{\mathbb{P}, \mathbb{Q}}^{(E, \Omega)}$ satisfying $(X, \Gamma) \leq (Y, \Delta)$, there exists an $L(X)$ -homomorphism $f_{(X, \Gamma)}^{(Y, \Delta)}$ of $\hat{A}_{(X, \Gamma)}(\phi_{(X, \Gamma)})$ into $\hat{A}_{(Y, \Delta)}(\phi_{(Y, \Delta)})$, i.e. L -homomorphism $f_{(X, \Gamma)}^{(Y, \Delta)}$ of $\hat{A}_{(X, \Gamma)}$ into $\hat{A}_{(Y, \Delta)}$ which maps $\phi_{(X, \Gamma)}(e)$ to $\phi_{(Y, \Delta)}(e)$ for each $e \in X$. These homomorphisms have the properties that $f_{(X, \Gamma)}^{(X, \Gamma)}$ is the identity mapping and that $f_{(Y, \Delta)}^{(Z, \Lambda)} f_{(X, \Gamma)}^{(Y, \Delta)} = f_{(X, \Gamma)}^{(Z, \Lambda)}$ if $(X, \Gamma) \leq (Y, \Delta) \leq (Z, \Lambda)$. Hence the pair of families $(\hat{A}_{(X, \Gamma)} \mid (X, \Gamma) \in M_{\mathbb{P}, \mathbb{Q}}^{(E, \Omega)})$ and $(f_{(X, \Gamma)}^{(Y, \Delta)} \mid (X, \Gamma), (Y, \Delta) \in M_{\mathbb{P}, \mathbb{Q}}^{(E, \Omega)} \text{ and } (X, \Gamma) \leq (Y, \Delta))$ forms a direct system, which

1) 2) In this expression, $\bigcup (X_\eta \mid \eta \leq \beta)$ denotes the subsequence of $(\xi \mid \xi < \alpha)$ which consists of all ordinals belonging to the set-union $\bigcup (X_\eta \mid \eta \leq \beta)$.

is called a direct system $(\mathcal{P}, \mathcal{Q})$ -naturally defined by $(E, \Omega; K)$.

The direct limit of a direct system $(\mathcal{P}, \mathcal{Q})$ -naturally defined by $(E, \Omega; K)$ is called a $(\mathcal{P}, \mathcal{Q})$ -natural limit structure with respect to $(E, \Omega; K)$ and denoted by $\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)$. Note that

$\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)$ is unique up to L -isomorphism if it exists. Now

we define a mapping ϕ of E into $D[\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)]$ as

$\phi(e) = \langle \phi_{(X, \Gamma)}(e), (X, \Gamma) \rangle$ for some $(X, \Gamma) \in M_{(\mathcal{P}, \mathcal{Q})}(E, \Omega)$ satisfying $X \ni e$, where $\langle \phi_{(X, \Gamma)}(e), (X, \Gamma) \rangle$ denotes the member of

$D[\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)]$ that contains $\langle \phi_{(X, \Gamma)}(e), (X, \Gamma) \rangle$. Of course, this is well defined, because if $(X, \Gamma) \leq (Y, \Delta)$ then

$$\langle \phi_{(X, \Gamma)}(e), (X, \Gamma) \rangle = \langle f_{(X, \Gamma)}^{(Y, \Delta)} \phi_{(X, \Gamma)}(e), (Y, \Delta) \rangle = \langle \phi_{(Y, \Delta)}(e), (Y, \Delta) \rangle.$$

The mapping ϕ defined as above is called a natural interpretation of E to $\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)$.

Under the above definitions and notation, we shall prove the

LEMMA 5. Suppose $\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)$ is in K . Then

$(\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K), \phi)$ is in $F(E, \Omega; K)$.

Proof. It is easily seen that $(\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K), \phi)$ is in $(E, \Omega; K)$ and $\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)$ is generated by $\{\phi(e) \mid e \in E\}$.

Now let (\mathbb{B}, ψ) be any member of $(E, \Omega; K)$. We shall prove that there exists an $L(E)$ -homomorphism of $\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)(\phi)$ into $\mathbb{B}(\psi)$.

Let θ be any atomic sentence of $L(E)$ which is valid in $\mathbb{L}_{(\mathcal{P}, \mathcal{Q})}(E, \Omega; K)(\phi)$. Then there exists some (X, Γ) in $M_{(\mathcal{P}, \mathcal{Q})}(E, \Omega)$ such that $\mathbb{A}_{(X, \Gamma)}(\phi_{(X, \Gamma)}) \models \theta$. Since (\mathbb{B}, ψ) is in $(E, \Omega; K)$, $(\mathbb{B}, \psi|X)^3$ is in $(X, \Gamma; K)$. Hence there exists an $L(X)$ -homomorphism of $\mathbb{A}_{(X, \Gamma)}(\phi_{(X, \Gamma)})$ into $\mathbb{B}(\psi|X)$, because $(\mathbb{A}_{(X, \Gamma)}, \phi_{(X, \Gamma)})$ is in $F(X, \Gamma; K)$. Hence we have $\mathbb{B}(\psi|X) \models \theta$,

3) $\psi|X$ denotes the mapping which is the restriction of ψ to X .

and hence $\mathbb{B}(\psi) \models \theta$. Therefore there exists an $L(E)$ -homomorphism of $\mathbb{L}_{\mathbb{P}, \mathbb{Q}}(E, \Omega; K)(\phi)$ into $\mathbb{B}(\psi)$. Hence $(\mathbb{L}_{\mathbb{P}, \mathbb{Q}}(E, \Omega; K), \phi)$ is in $F(E, \Omega; K)$. This completes the proof.

LEMMA 6. Let K be a class of structures for L such that for any $L(\mathbb{m}, \mathbb{n})$ -defining pair (X, Γ) , $(X, \Gamma; K) \neq \emptyset$ implies $F(X, \Gamma; K) \neq \emptyset$. And let Σ be the set of all generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentences that are valid in all structures in K . Then the following assertions hold for any $L(\mathbb{m}, \mathbb{n})$ -defining pair (E, Ω) :

- (1) $F(E, \Omega; \Sigma^*) \neq \emptyset$ if and only if $F(E, \Omega; K) \neq \emptyset$.
- (2) If $(\mathbb{F}, \phi) \in F(E, \Omega; K)$ and $(\mathbb{G}, \psi) \in F(E, \Omega; \Sigma^*)$, then $\mathbb{F}(\phi)$ and $\mathbb{G}(\psi)$ are $L(E)$ -isomorphic.

Proof. Let $E = \{e_\xi \mid \xi < \alpha\}$ and let $\Omega = \{\theta_\eta(e_\xi \mid \xi < \alpha) \mid \eta < \beta\}$ ⁴⁾, where $\alpha < \omega_{\mathbb{m}}$ and $\beta < \omega_{\mathbb{n}}$.

First we shall prove the assertion (1). Assume that $F(E, \Omega; \Sigma^*) = \emptyset$. Then by Lemmas 1, 2 and the Criterion, $(E, \Omega; \Sigma^*) = \emptyset$. Hence $(E, \Omega; K) = \emptyset$, because $(E, \Omega; K) \subseteq (E, \Omega; \Sigma^*)$. Hence we have $F(E, \Omega; K) = \emptyset$. Conversely assume that $F(E, \Omega; K) = \emptyset$. Then by the assumption of this lemma, $(E, \Omega; K) = \emptyset$. Hence for any $(\mathbb{A}, \theta) \in (E, \emptyset; K)$,

$$\mathbb{A}(\theta) \models \bigvee (\neg \theta_\eta(e_\xi \mid \xi < \alpha) \mid \eta < \beta).$$

And hence for every structure \mathbb{A} in K ,

$$\mathbb{A} \models \bigvee (x_\xi \mid \xi < \alpha) [\bigvee (\neg \theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta)].$$

4) We denote by $\theta_\eta(e_\xi \mid \xi < \alpha)$ the atomic sentence of $L(E)$ which is obtained from an atomic formula $\theta_\eta(x_\xi \mid \xi < \alpha)$ of L by replacing the variables x_ξ by the constant symbols e_ξ respectively. Note that any atomic sentence of $L(E)$ can be written in such a form.

Therefore the generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentence

$$\forall(x_\xi \mid \xi < \alpha)[\forall(\neg\theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta)]$$

belongs to Σ . Hence $(E, \Omega; \Sigma^*) = \emptyset$, and hence $F(E, \Omega; \Sigma^*) = \emptyset$.

Next we shall prove the assertion (2). Assume that $(\mathbb{F}, \phi) \in F(E, \Omega; K)$ and $(\mathbb{G}, \psi) \in F(E, \Omega; \Sigma^*)$. Since $(\mathbb{F}, \phi) \in (E, \Omega; K) \subseteq (E, \Omega; \Sigma^*)$, there exists an $L(E)$ -homomorphism h of $\mathbb{G}(\psi)$ onto $\mathbb{F}(\phi)$. Now let $\theta(e_\xi \mid \xi < \alpha)$ be any atomic sentence of $L(E)$ such that $\mathbb{F}(\phi) \models \theta(e_\xi \mid \xi < \alpha)$. Then, for any $(\mathbb{A}, \theta) \in (E, \Omega; K)$, we have

$$\mathbb{A}(\theta) \models \theta(e_\xi \mid \xi < \alpha),$$

because there exists an $L(E)$ -homomorphism of $\mathbb{F}(\phi)$ into $\mathbb{A}(\theta)$.

Hence for any $(\mathbb{B}, \tau) \in (E, \emptyset; K)$,

$$\mathbb{B}(\tau) \models \bigwedge(\theta_\eta(e_\xi \mid \xi < \alpha) \mid \eta < \beta) \rightarrow \theta(e_\xi \mid \xi < \alpha).$$

And hence for every $\mathbb{B} \in K$,

$$\mathbb{B} \models \forall(x_\xi \mid \xi < \alpha)[\bigwedge(\theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta) \rightarrow \theta(x_\xi \mid \xi < \alpha)].$$

Therefore the $L(\mathbb{m}, \mathbb{n})$ -implicational sentence

$$\forall(x_\xi \mid \xi < \alpha)[\bigwedge(\theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta) \rightarrow \theta(x_\xi \mid \xi < \alpha)]$$

belongs to Σ . Since $\mathbb{G} \in \Sigma^*$ and $\mathbb{G}(\psi) \models \theta_\eta(e_\xi \mid \xi < \alpha)$ for all $\eta < \beta$, we have

$$\mathbb{G}(\psi) \models \theta(e_\xi \mid \xi < \alpha).$$

Hence the $L(E)$ -homomorphism h of $\mathbb{G}(\psi)$ onto $\mathbb{F}(\phi)$ is an $L(E)$ -isomorphism. This completes the proof.

The following lemma can be easily obtained from the above lemma and the definition of an (\mathbb{m}, \mathbb{n}) -natural limit structure.

LEMMA 7. Let K and Σ be the same as in Lemma 6. Then the following assertions hold for any L -defining pair (E, Ω) :

- (1) $\mathbb{L}_{(\mathbb{m}, \mathbb{n})}(E, \Omega; K)$ exists if and only if $\mathbb{L}_{(\mathbb{m}, \mathbb{n})}(E, \Omega; \Sigma^*)$ exists.

- (2) $\underline{L}_{(\underline{m}, \underline{n})}(E, \Omega; K)$ and $\underline{L}_{(\underline{m}, \underline{n})}(E, \Omega; \Sigma^*)$ are L -isomorphic if both exist.

§ 4. Main theorems.

Throughout this section, we assume that L is a first order language with equality and with an infinite set $\{v_\xi \mid \xi < \omega_{(\underline{m})}\}$ of (\underline{m}) variables as in the preceding sections.

THEOREM 1. Assume that (\underline{m}) and (\underline{n}) are regular infinite cardinals, and let K be any class of structures for L . Then $\underline{U}_{(\underline{m}, \underline{n})} \text{IL}_{(\underline{n})} \text{SP}(K)$ is the least generalized $L_{(\underline{m}, \underline{n})}$ -implicational class containing K . That is, if Σ is the set of all generalized $L_{(\underline{m}, \underline{n})}$ -implicational sentences that are valid in all structures in K , then

$$\Sigma^* = \underline{U}_{(\underline{m}, \underline{n})} \text{IL}_{(\underline{n})} \text{SP}(K).$$

Proof. By Lemmas 1, 2, 3, and 4, it is clear that

$$\Sigma^* \supseteq \underline{U}_{(\underline{m}, \underline{n})} \text{IL}_{(\underline{n})} \text{SP}(K).$$

We shall prove that

$$\Sigma^* \subseteq \underline{U}_{(\underline{m}, \underline{n})} \text{IL}_{(\underline{n})} \text{SP}(K).$$

Assume that (A) is any structure in Σ^* . Now let M be the set of all non-empty subsets of $D[(A)]$ whose cardinals are less than (\underline{m}) . Since (\underline{m}) is regular, M forms an (\underline{m}) -directed partially ordered set under the inclusion relation. For each $\mu \in M$, let $(A)_\mu$ be the substructure of (A) generated by μ . Then $((A)_\mu \mid \mu \in M)$ forms an (\underline{m}) -directed family of structures, and clearly

$$(A) = \bigcup_{\mu \in M} (A)_\mu.$$

Hence, in order to prove $\Sigma^* \subseteq \underline{U}_{(\underline{m}, \underline{n})} \text{IL}_{(\underline{n})} \text{SP}(K)$, it suffices to prove that each $(A)_\mu$ is in $\text{IL}_{(\underline{n})} \text{SP}(K)$.

By Lemma 1, each $(A)_\mu$ is in Σ^* . Therefore we have

$$((A)_\mu, \psi_\mu) \in F(E_\mu, \Omega_\mu; \Sigma^*),$$

where $\bar{E}_\mu = \bar{\mu}$, ψ_μ is a one-to-one mapping of E_μ onto μ , and Ω_μ is the set of all atomic sentences of $L(E_\mu)$ which are valid in (A_μ, ψ_μ) . Hence for any $L(\mathbb{m}, \mathbb{n})$ -defining pair $(X, \Gamma) \in M_{\mathbb{m}, \mathbb{n}}(E_\mu, \Omega_\mu)$, $(A_\mu, \psi_\mu|X)$ is in $(X, \Gamma; \Sigma^*)$, and hence $F(X, \Gamma; \Sigma^*) \neq \emptyset$ follows from Lemmas 1, 2 and the Criterion.

Therefore there exists a direct system (\mathbb{m}, \mathbb{n}) -naturally defined by $(E_\mu, \Omega_\mu; \Sigma^*)$, which is an \mathbb{n} -direct system consisting of structures in Σ^* , because $\bar{E}_\mu < \mathbb{m}$ and \mathbb{n} is regular. Hence the (\mathbb{m}, \mathbb{n}) -natural limit structure $L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; \Sigma^*)$ exists, and by Lemma 4, it is in Σ^* . Therefore by Lemma 5, we have

$$(L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; \Sigma^*), \phi_\mu) \in F(E_\mu, \Omega_\mu; \Sigma^*),$$

where ϕ_μ is the natural interpretation of E_μ to

$L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; \Sigma^*)$. Hence we have that A_μ and $L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; \Sigma^*)$ are L-isomorphic, that is,

$$A_\mu \cong_L L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; \Sigma^*).$$

Since $SSP(K) \subseteq ISP(K)$ and $PSP(K) \subseteq ISP(K)$, it follows from the Criterion that for any $L(\mathbb{m}, \mathbb{n})$ -defining pair (Y, Δ) , $(Y, \Delta; SP(K)) \neq \emptyset$ implies $F(Y, \Delta; SP(K)) \neq \emptyset$. Moreover Σ can be considered as the set of all generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentences that are valid in all structures in $SP(K)$, because $K \subseteq SP(K) \subseteq \Sigma^*$ follows from Lemmas 1 and 2. Hence by Lemma 7, $L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; SP(K))$ exists, and it is L-isomorphic to $L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; \Sigma^*)$. Therefore we have

$$A_\mu \cong_L L_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu; SP(K)).$$

Since $M_{(\mathbb{m}, \mathbb{n})}(E_\mu, \Omega_\mu)$ is \mathbb{n} -directed, we have that each A_μ is L-isomorphic to a direct limit of an \mathbb{n} -direct system consisting of structures in $SP(K)$, i.e. $A_\mu \in IL_{\mathbb{n}} SP(K)$, as desired. This completes the proof.

THEOREM 2. Assume that \aleph is a regular infinite cardinal not greater than the cardinal \aleph , and let K be any class of structures for L . Then $IL_{\aleph}SP(K)$ is the least generalized $L(\aleph, \aleph)$ -implicational class containing K . That is, if Σ is the set of all generalized $L(\aleph, \aleph)$ -implicational sentences that are valid in all structures in K , then

$$\Sigma^* = IL_{\aleph}SP(K).$$

Note that if $\aleph \geq \aleph$, then every generalized $L(\aleph, \aleph)$ -implicational sentence is equivalent to a generalized $L(\aleph, \aleph)$ -implicational sentence.

Proof. By Lemmas 1, 2, and 4, it is clear that

$$\Sigma^* \supseteq IL_{\aleph}SP(K).$$

We shall prove that

$$\Sigma^* \subseteq IL_{\aleph}SP(K).$$

Assume that \mathcal{A} is any structure in Σ^* . Then we have

$$(\mathcal{A}, \psi) \in F(E, \Omega; \Sigma^*),$$

where $\bar{E} = \overline{D[\mathcal{A}]}$, ψ is a one-to-one mapping of E onto $D[\mathcal{A}]$, and Ω is the set of all atomic sentences of $L(E)$ that are valid in $\mathcal{A}(\psi)$. Hence for any $L(\aleph, \aleph)$ -defining pair $(X, \Gamma) \in M_{\aleph, \aleph}(E, \Omega)$, $(\mathcal{A}, \psi \upharpoonright X)$ is in $(X, \Gamma; \Sigma^*)$, and hence $F(X, \Gamma; \Sigma^*) \neq \emptyset$ follows from Lemmas 1, 2, and the Criterion. Therefore there exists a direct system (\aleph, \aleph) -naturally defined by $(E, \Omega; \Sigma^*)$, which is an \aleph -direct system consisting of structures in Σ^* , because \aleph is regular. Hence the (\aleph, \aleph) -natural limit structure $L_{\aleph, \aleph}(E, \Omega; \Sigma^*)$ exists, and by Lemma 4 it is in Σ^* . Therefore by Lemma 5, we have

$$(L_{\aleph, \aleph}(E, \Omega; \Sigma^*), \phi) \in F(E, \Omega; \Sigma^*),$$

where ϕ is the natural interpretation of E to $L_{\aleph, \aleph}(E, \Omega; \Sigma^*)$.

Hence we have

$$\mathbb{A} \cong_L \mathbb{L}_{\mathbb{n}, \mathbb{n}}(E, \Omega; \Sigma^*).$$

On the other hand, for any $L(\mathbb{n}, \mathbb{n})$ -defining pair (Y, Δ) , $(Y, \Delta; SP(K)) \neq \emptyset$ implies $F(Y, \Delta; SP(K)) \neq \emptyset$. Moreover Σ^* can be considered as the class defined by the set of all generalized $L(\mathbb{n}, \mathbb{n})$ -implicational sentences that hold in $SP(K)$.

Hence by Lemma 7, $\mathbb{L}_{\mathbb{n}, \mathbb{n}}(E, \Omega; SP(K))$ exists and it is L -isomorphic to $\mathbb{L}_{\mathbb{n}, \mathbb{n}}(E, \Omega; \Sigma^*)$. Therefore we have

$$\mathbb{A} \cong_L \mathbb{L}_{\mathbb{n}, \mathbb{n}}(E, \Omega; SP(K)).$$

This implies that $\mathbb{A} \in IL_{\mathbb{n}} SP(K)$, because $M_{\mathbb{n}, \mathbb{n}}(E, \Omega)$ is \mathbb{n} -directed. Therefore we have $\Sigma^* \subseteq IL_{\mathbb{n}} SP(K)$. This completes the proof.

We denote by $A(L)$ the set of all atomic formulas of the language L .

THEOREM 3. Assume that the infinite cardinal \mathbb{m} is regular and \mathbb{n} is any cardinal $> \overline{A(L)}$, and let K be any class of structures for L . Then $\bigcup_{\mathbb{m}} ISP(K)$ is the least generalized $L(\mathbb{m}, \mathbb{n})$ -implicational class containing K . That is, if Σ is the set of all generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentences that are valid in all structures in K , then

$$\Sigma^* = \bigcup_{\mathbb{m}} ISP(K).$$

Proof. By Lemmas 1, 2, and 3, it is clear that

$$\Sigma^* \supseteq \bigcup_{\mathbb{m}} ISP(K).$$

We shall prove that

$$\Sigma^* \subseteq \bigcup_{\mathbb{m}} ISP(K).$$

Assume that \mathbb{A} is any structure in Σ^* . Now let M be the set of all non-empty subsets of $D[\mathbb{A}]$ whose cardinals are less than \mathbb{m} . Then M forms an \mathbb{m} -directed partially ordered set under the

inclusion relation, because \mathbb{m} is regular. For each $\mu \in M$, let \mathbb{A}_μ be the substructure of \mathbb{A} generated by μ . Then $(\mathbb{A}_\mu \mid \mu \in M)$ forms an \mathbb{m} -directed family of structures, and clearly

$$\mathbb{A} = \bigcup (\mathbb{A}_\mu \mid \mu \in M).$$

By Lemma 1, each \mathbb{A}_μ is in Σ^* . Hence we have

$$(\mathbb{A}_\mu, \psi_\mu) \in F(E_\mu, \Omega_\mu; \Sigma^*),$$

where $\bar{E}_\mu = \bar{\mu}$, ψ_μ is a one-to-one mapping of E_μ onto μ , and Ω_μ is the set of all atomic sentences of $L(E_\mu)$ that are valid in $\mathbb{A}_\mu(\psi_\mu)$. On the other hand, for any $L(\mathbb{m}, \mathbb{n})$ -defining pair (Y, Δ) , $(Y, \Delta; SP(K)) \neq \emptyset$ implies $F(Y, \Delta; SP(K)) \neq \emptyset$. Moreover Σ can be considered as the set of all generalized $L(\mathbb{m}, \mathbb{n})$ -implicational sentences that hold in $SP(K)$. Hence by (1) of Lemma 6, we have $F(E_\mu, \Omega_\mu; SP(K)) \neq \emptyset$, because $F(E_\mu, \Omega_\mu; \Sigma^*) \neq \emptyset$ and (E_μ, Ω_μ) is an $L(\mathbb{m}, \mathbb{n})$ -defining pair. Now take

$$(\mathbb{B}_\mu, \phi_\mu) \in F(E_\mu, \Omega_\mu; SP(K)).$$

Then by (2) of Lemma 6, $\mathbb{A}_\mu(\psi_\mu)$ and $\mathbb{B}_\mu(\phi_\mu)$ are $L(E_\mu)$ -isomorphic. Hence $\mathbb{A}_\mu \in ISP(K)$, and hence $\mathbb{A} \in U_{\mathbb{m}} ISP(K)$. Therefore we have $\Sigma^* \subseteq U_{\mathbb{m}} ISP(K)$. This completes the proof.

As immediate consequences of Theorems 1, 2, and 3, we have the following characterizations of generalized $L(\mathbb{m}, \mathbb{n})$ -implicational classes respectively:

COROLLARY 1. Assume that \mathbb{m} and \mathbb{n} are regular infinite cardinals. Then, a class K of structures for L is a generalized $L(\mathbb{m}, \mathbb{n})$ -implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P(K) \subseteq K$, $U_{\mathbb{m}}(K) \subseteq K$, and $L_{\mathbb{n}}(K) \subseteq K$.

COROLLARY 2. Assume that \mathbb{n} is a regular infinite cardinal

not greater than the cardinal \mathfrak{m} . Then, a class K of structures for L is a generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P(K) \subseteq K$, and $L_{\mathfrak{n}}(K) \subseteq K$.

COROLLARY 3. Assume that the infinite cardinal \mathfrak{m} is regular and \mathfrak{n} is any cardinal $> \overline{A(L)}$. Then, a class K of structures for L is a generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P(K) \subseteq K$, and $U_{\mathfrak{m}}(K) \subseteq K$.

Remarks on $L(\mathfrak{m}, \mathfrak{n})$ -implicational classes. From Theorem 1, we can easily obtain the following analogous theorem for $L(\mathfrak{m}, \mathfrak{n})$ -implicational classes:

(I) Assume that \mathfrak{m} and \mathfrak{n} are regular infinite cardinals, and let K be any class of structures for L . Then $U_{\mathfrak{m}-\mathfrak{n}}^{IL, SP^*}(K)$ is the least $L(\mathfrak{m}, \mathfrak{n})$ -implicational class containing K .

We simply explain this fact. Let Σ be the set of all $L(\mathfrak{m}, \mathfrak{n})$ -implicational sentences valid in all structures in K , and let Γ be the set of all generalized $L(\mathfrak{m}, \mathfrak{n})$ -implicational sentences valid in all structures in $K \cup \{E\}$, where E is a L -trivial structure. Then it is clear that $\Sigma^* = \Gamma^*$ and $IP^*(K) = IP(K \cup \{E\})$. Hence by Theorem 1, we have

$$\begin{aligned} U_{\mathfrak{m}-\mathfrak{n}}^{IL, SP^*}(K) &= U_{\mathfrak{m}-\mathfrak{n}}^{IL, SIP^*}(K) \\ &= U_{\mathfrak{m}-\mathfrak{n}}^{IL, SIP}(K \cup \{E\}) = U_{\mathfrak{m}-\mathfrak{n}}^{IL, SP}(K \cup \{E\}) = \Gamma^* = \Sigma^*. \end{aligned}$$

Hence $U_{\mathfrak{m}-\mathfrak{n}}^{IL, SP^*}(K)$ is the least $L(\mathfrak{m}, \mathfrak{n})$ -implicational class containing K .

By the similar method as in the above, we can obtain the following theorems (II) and (III) analogous to Theorems 2 and 3 respectively.

(II) Assume that \mathfrak{n} is a regular infinite cardinal not greater than the cardinal \mathfrak{m} , and let K be any class of

structures for L . Then $IL_{\mathbb{N}}SP^*(K)$ is the least $L(\mathbb{M}, \mathbb{N})$ -implicational class containing K .

(III) Assume that the infinite cardinal \mathbb{M} is regular and \mathbb{N} is any cardinal $> \overline{A(L)}$, and let K be any class of structures for L . Then $U_{\mathbb{M}}ISP^*(K)$ is the least $L(\mathbb{M}, \mathbb{N})$ -implicational class containing K .

The following characterizations of $L(\mathbb{M}, \mathbb{N})$ -implicational classes are immediately obtained from the theorems (I), (II), and (III) respectively.

(i) Assume that \mathbb{M} and \mathbb{N} are regular infinite cardinals. Then, a class K of structures for L is an $L(\mathbb{M}, \mathbb{N})$ -implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P^*(K) \subseteq K$, $U_{\mathbb{M}}(K) \subseteq K$ and $L_{\mathbb{N}}(K) \subseteq K$.

(ii) Assume that \mathbb{N} is a regular infinite cardinal not greater than the cardinal \mathbb{M} . Then, a class K of structures for L is an $L(\mathbb{M}, \mathbb{N})$ -implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P^*(K) \subseteq K$, and $L_{\mathbb{N}}(K) \subseteq K$.

(iii) Assume that the infinite cardinal \mathbb{M} is regular and \mathbb{N} is any cardinal $> \overline{A(L)}$. Then, a class K of structures for L is an $L(\mathbb{M}, \mathbb{N})$ -implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P^*(K) \subseteq K$, and $U_{\mathbb{M}}(K) \subseteq K$.

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